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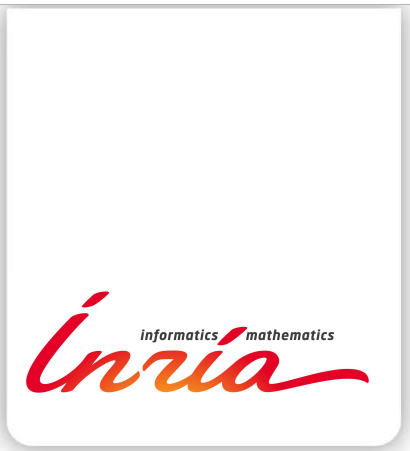

Competitive exclusion principle for SIS and SIR models with n strains

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Abstract: We consider SIS and SIR models with standard mass action and varying population, with n different pathogen strains of an infectious disease. We also consider the same models with vertical transmission. We prove that under generic conditions a competitive exclusion principle holds. To each strain a basic reproduction ratio can be associated. It corresponds to the case where only this strain exists. The basic reproduction ratio of the complete system is the maximum of each individual basic reproduction ratio.
Actually we also define an equivalent threshold for each strain. The winner of the competition is the strain with the maximum threshold. It turns out that this strain is the most virulent, i.e., this is the strain for which the endemic equilibrium gives the minimum population for the susceptible host population. This can be interpreted as a pessimization principle.

Key-words: Nonlinear dynamical systems, global stability, Lyapunov methods, competition, boundary equilibria

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Principe de compétition exclusive pour les modèles SIS et SIR avec n souches

Résumé : On considère les modèles SIS et SIR avec la loi de l'action de masse standard et à population non constante, avec n souches de pathogénes d'une maladie infectieuse. On considère aussi les mêmes modèles avec la transmission verticale. Nous prouvons sous des conditions génériques que le principe de compétition exclusive est vérifié. A chaque souche est associée un nombre de reproduction de base. Il correspond au cas où seule la souche correspondante existe. Le nombre de reproduction de base du système complet est le maximum des nombres de reproduction de base pris individuellement. En fait nous définissons aussi un seuil équivalent pour chaque souche. La souche gagnante de la compétition est celle qui maximise le seuil. Il se trouve aussi que c'est la souche la plus virulente, c'est-à-dire la souche pour laquelle l'équilibre endémique donne le minimum des susceptibles dans la population hôte. Ceci peut être interprété comme un principe de pessimization.

Mots-clés : Sysèmes dynamiques non-linéaires, stabilité globale, méthodes de Lyapunov, compétition, équilibres frontières

1 Introduction

One of the most famous principles in theoretical ecology is the competitive exclusive principle, sometimes called Gause's Law of competitive exclusion, that stipulates: *Two species competing for the same resources cannot coexist indefinitely on the same ecological niche*[14, 23, 26, 29]. However, according to Amstrong and McGehee [3], Volterra [35] was the first to use mathematical model to suggest that the indefinite coexistence of two or more species limited by the same resource is impossible. An abundance of literature was dedicated to the validation of this principle to epidemiological and/or host-parasite models [1, 2, 3, 4, 6, 7, 8, 14, 16, 17, 22, 23, 26, 27, 28, 29] and the references therein.

In mathematical epidemiology the seminal paper [6] proves a competitive exclusion of infection with different levels in single host populations. When only one strain is present a basic reproduction ratio can be computed [12, 33]. Hence to each strain is associated such a number. The authors show that the competitive exclusion principle holds generically and that the winning strain is the one which maximizes its basic reproduction number. More precisely it is proven that all but one strain disappear, the winning strain being persistent. The model [6] considered by Bremermann and Thieme is a SIS model with n strains.

Ackeh and Allen [1] consider a SIR model with n strains and vertical transmission. They generalize a two-pathogen study by Andreasen and Pugliese [5]. They assume for the population dynamics a constant birth rate and a nonlinear death rate. Globally the population dynamics, without disease, is a generalized logistic type. They also assume a vertical transmission. The analysis is done for mass action incidence. The authors in [2] consider the SIS and SIR models with standard incidence. In both cases they derive sufficient conditions for competitive exclusion between the n strains. One of these conditions involved the maximization of the basic reproduction ratio for the winning strain. As in the paper of Bremermann and Thieme this is a persistence result.

In [18], an analogous system has been considered with a general recruitment function $f(N)$, but without vertical transmission. Similar competitive exclusion results have been obtained (Theorem 3.3) under some additional assumptions. The stability of the endemic equilibria is obtained but unfortunately, as stated by the authors, the assumed hypotheses are generally hard to check. The study of the reduced system is still under investigation.

Recently attention has focused on understanding the factors that lead to coexistence or to competitive exclusion [10, 9, 31]. In the absence of multiple infections and the presence of complete cross-immunity, only the parasite strain persists that has the maximal basic reproduction ratio. See the survey [31]. This result has been proved under quite a few restrictions. However the validity of such a principle has been challenged. For example when the infection is transmitted both vertically and horizontally, simulations have shown that strains with lower virulence can outcompete strains with higher basic reproduction ratio [25].

To continue to explore this issue we consider a SIS or a SIR model with vertical and horizontal transmission and a different population dynamics. In our SIR model we, more generally, assume that when an individual recovers then only a fraction is cross-immunized. The other part of the individuals are again susceptible after recovering. We consider the bilinear mass action incidence. We assume, when acquired, total cross immunity and no superinfection. We prove that under generic conditions the principle of competitive exclusion applies. Compared to the other results this is not the basic reproduction ratio which determines the fate of the strain. We associate to each i strain a threshold $\mathcal{T}_{0,i}$. This a threshold since $\mathcal{T}_{0,i} > 1$ is equivalent to $\mathcal{R}_{0,i} > 1$. The strain which maximizes its threshold wins the competition. Hence we can speak of a maximization of a threshold but not maximation of the basic reproduction ratio. We prove that an endemic equilibrium exists and that this equilibrium is globally asymptotically stable.

Actually the winner is the strain which minimizes the number of susceptibles at the equilibrium, which is not necessarily the strain which maximizes its basic reproduction ratio. The same kind of phenomenon is observed in [17]. This can be seen as a pessimization principle [11, 30].

Most authors [6, 1, 2] prove the competitive exclusion by persistence : under some conditions, all densities of infected strains, but one tend to 0. The remaining strain stays positive, i.e., is persistent.

We consider in this paper, models with a different population dynamics. We consider SIS or SIR models with horizontal and vertical transmission. We consider a constant recruitment (or immigration of susceptible individuals) and constant death rate, which gives the dynamics without disease $\dot{N} = \Lambda - \mu\, N$.

Our law is less general but we can obtain more precise results. We always have a unique disease free equilibrium (DFE) and when the basic reproduction ratio is greater or equal to one some boundary endemic equilibrium exists. Under a generic condition we prove that there is an equilibrium in a face of the nonnegative orthant, corresponding to the extinction of all the strains but one, which is globally asymptotically stable in the interior of the orthant and the corresponding face. We also describe the stability on each face. The evolution in a face of the orthant biologically corresponds to the nonexistence of some strains.

In this paper we prove this result by use of Lyapunov functions. Actually, the function we use is a “Volterra" like Lyapunov function. This kind of function has now been used successfully to ascertain global stability results in epidemiological models. See for example the references in [13, 32].

The paper is organized as follows :

In the next section we introduce the special SIS model, with vertical transmission. We show that, to obtain general results, we can however reduce the study to a model with peculiar assumptions on the parameters. In this section we compute the equilibria and the basic reproduction ratio. We have also defined an equivalent threshold.

Section 3 is devoted to the study of stability, under non-stringent generic hypothesis. We obtain global stability results on the biological domain.The last section examines the case of a SIR model with vertical transmission.

2 SIS model with vertical transmission

2.1 The model

We assume that there is n strains of pathogens. The host population is divided into susceptible, S, individuals infected with strain i, I_i. There is a mass action horizontal transmission. A fraction of the newborn, coming from the infectious individual, is infectious. In other words there is a vertical transmission. A fraction p_i of the offspring from the infected class I_i is infected at birth. Hence, $p_i\, I_i$ individuals enters into infected class I_i and the same quantity is lacking from recruitment. When recovered individuals are again susceptible to the disease.

We have the following diagram:

The model has the form

$$\begin{cases} \dot{S} = \Lambda - \displaystyle\sum_{i=1}^{n} \beta_i\, S\, I_i + \sum_{i=1}^{n} (\gamma_i - p_i)\, I_i - \mu\, S, \\ \\ \dot{I}_i = \beta_i\, S\, I_i - (\mu + \delta_i + \gamma_i - p_i)\, I_i, \text{ for } i = 1, \ldots, n. \end{cases} \tag{1}$$

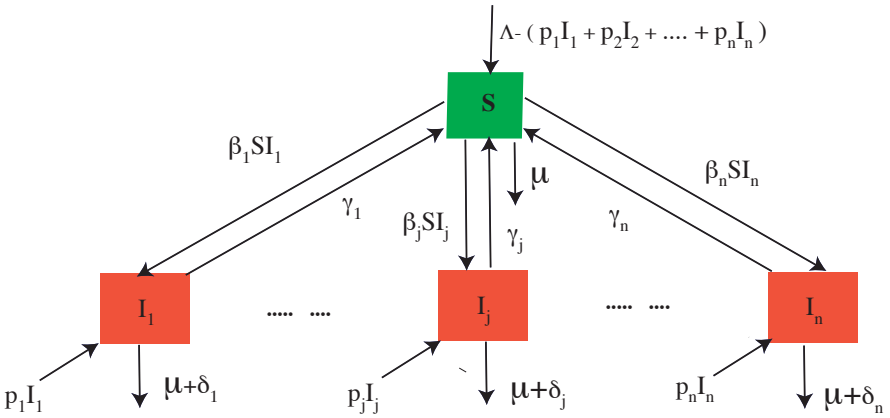

Figure 1: A SIS model with n strains

The different parameters of our model are definite as:

Λ	:	Recruitement of the susceptible individuals (birth, ...)
β_i	:	transmission coefficient by i-th strain
μ	:	Natural mortality rate
δ_i	:	additional mortality of i-th strain,
γ_i	:	recovered rate of i-th strain.
p_i	:	rate at which an infectious host (by i-th strain) gives birth to a new infected host

2.2 Reduction to an equivalent system

In the section we will show that, without reducing the generality of our study, we can assume that for any index $i = 1, \cdots, n$ we have $\gamma_i \geq p_i$. If this is not the case we denote by $\mathcal{I}$ the set of indices defined by

$$\mathcal{I} = \{ i = 1, \cdots, n \ \mid \ p_i > \gamma_i \ \}$$

If $\mathcal{I} \neq \emptyset$, we consider the following change of variables

$$P = S - \sum_{j \in I} \frac{\gamma_j - p_j}{\beta_j}$$

In this new coordinates the system, for his first equation, becomes

$$\dot{P} = \Lambda + \mu \sum_{j \in \mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} - \sum_{i=1}^{n} \beta_i \, P \, I_i + \sum_{i=1}^{n} \left[\beta_i \sum_{j \in \mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} + \gamma_i - p_i \right] I_i - \mu \, P$$

This can be expressed as

$$\dot{P} = \Lambda + \mu \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} - \sum_{i=1}^{n} \beta_i P I_i + \sum_{i\in\mathcal{I}} \left[\beta_i \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} + \gamma_i - p_i \right] I_i + \sum_{i\notin\mathcal{I}} \left[\beta_i \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} + \gamma_i - p_i \right] I_i - \mu P$$

or equivalently

$$\dot{P} = \Lambda + \mu \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} - \sum_{i=1}^{n} \beta_i P I_i + \sum_{i\in\mathcal{I}} \left[\beta_i \sum_{\substack{j\in\mathcal{I} \\ j\neq i}} \frac{p_j - \gamma_j}{\beta_j} \right] I_i + \sum_{i\notin\mathcal{I}} \left[\beta_i \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} + \gamma_i - p_i \right] I_i - \mu P$$

We define

$$\tilde{\gamma}_i = \begin{cases} \beta_i \displaystyle\sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j} + \gamma_i & \text{if } i \neq \mathcal{I} \\ \beta_i \displaystyle\sum_{\substack{j\in\mathcal{I} \\ j\neq i}} \frac{p_j - \gamma_j}{\beta_j} + p_i & \text{if } i \in \mathcal{I} \end{cases}$$

and

$$\tilde{\Lambda} = \Lambda + \mu \sum_{j\in\mathcal{I}} \frac{p_j - \gamma_j}{\beta_j}$$

With these definitions we have

$$\dot{P} = \tilde{\Lambda} - \sum_{i=1}^{n} \beta_i P I_i + \sum_{i=1}^{n} (\tilde{\gamma}_i - p_i) I_i - \mu P,$$

and for $i = 1, \cdots, n$

$$\dot{I}_i = \beta_i P I_i - (\mu + \delta_i + \tilde{\gamma}_i - p_i) I_i.$$

with $\tilde{\Lambda} \geq 0$ and $\tilde{\gamma}_i \geq p_i$.

We have proved that, by a change of variable, we can always assume each of the parameters system (1) satisfies, for each index, $\gamma_i \geq p_i$. Then we will assume, without loss of generality, the following hypothesis in the sequel

$$\mathbf{H} \qquad \text{For } i = 1, \cdots, n \quad \gamma_i \geq p_i \tag{2}$$

2.3 Equilibria

We denote by $N = S + \sum_{i=1}^{n} I_i$ the total host population. The evolution of N is given by

$$\dot{N} = \Lambda - \mu N - \sum_{i=1}^{n} \delta_i I_i \leq \Lambda - \mu N.$$

The region defined by

$$\Omega = \left\{ (S, I) \in \mathbb{R}_+^{n+1} \;\middle|\; S + \sum_{i=1}^{n} I_i \leq \frac{\Lambda}{\mu} \right\}$$

is a compact attracting positively invariant set for system (1).

The disease free equilibrium (DFE) is given by $(S^*, 0, \ldots, 0)$ with $S^* = \frac{\Lambda}{\mu}$. This equilibrium belongs to Ω.

In the presence of only the i strain the model is a two dimensional model (with S and I_i). The basic reproduction ratio [33, 12] of this model, with vertical transmission, is given by

$$\mathcal{R}_{0,i} = \frac{\beta_i \dfrac{\Lambda}{\mu} + p_i}{(\mu + \delta_i + \gamma_i)}.$$

The system (1) has n endemic equilibria, located in the boundary of the nonnegative orthant. Namely $(\bar{S}_1, \bar{I}_1, 0, \ldots, 0)$, $(\bar{S}_2, 0, \bar{I}_2, 0, \ldots, 0)$, $\ldots$, $(\bar{S}_n, 0, \ldots, 0, \bar{I}_n)$, where

$$\bar{S}_i = \frac{\mu + \gamma_i + \delta_i - p_i}{\beta_i}, \quad \text{and} \quad \bar{I}_i = \frac{\Lambda}{\mu + \delta_i} \left(1 - \frac{\mu(\mu + \gamma_i + \delta_i - p_i)}{\beta_i \Lambda} \right).$$

The boundary equilibrium $(\bar{S}_i, 0, \ldots, \bar{I}_i, \ldots, 0)$ is in Ω if and only if

$$\mathcal{T}_{0,i} = \frac{\beta_i \, \Lambda}{\mu \, (\mu + \delta_i + \gamma_i - p_i)} > 1.$$

It is clear, with our running hypothesis (2), that

$$\mathcal{T}_{0,i} > 1 \Longleftrightarrow \mathcal{R}_{0,i} > 1.$$

Hence $\mathcal{T}_{0,i}$ is a threshold, in the sense of [19], and

$$\bar{S}_i = \frac{S^*}{\mathcal{T}_{0,i}}. \tag{3}$$

When $\mathcal{T}_{0,i} = 1$ then $\bar{S}_i = S^*$.

A condition for a coexistence between strain i and strain j will be

$$\frac{\mu + \gamma_i + \delta_i - p_i}{\beta_i} = \frac{\mu + \gamma_j + \delta_j - p_j}{\beta_i}.$$

or equivalently

$$\mathcal{T}_{0,i} = \mathcal{T}_{0,j} > 1$$

To have a coexistence in the interior of the nonnegative orthant will require n preceding equalities. This condition is clearly non generic.

The coexistence can occur if the death rate is density dependent [1, 5] or even with a standard incidence law and exponentially growing host populations [24].

3 Global stability analysis

Let us recall that we assume, without loss of generality, our standing hypothesis (2).

3.1 Global stability of DFE

Theorem 3.1 *If $\mathcal{T}_0 \leq 1$, the DFE is globally asymptotically stable in the nonnegative orthant. If $\mathcal{T}_0 > 1$, the DFE is unstable.*

Proof
We consider the LaSalle-Lyapunov function

$$V = \sum_{i=1}^{n} I_i.$$

Actually this is a nonnegative function, but nonpositive definite. Then we will use the LaSalle invariance principle on compact sets [15, 21, 20] to prove the asymptotic stability. By hypothesis each $\mathcal{T}_{0,i}$ satisfies $\mathcal{T}_{0,i} \leq 1$.

Computing the derivatives of V along the trajectories gives

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \left(\beta_i S - (\mu + \delta_i + \gamma_i - p_i)\right) I_i, \\
&\leq \sum_{i=1}^{n} \left(\beta_i \frac{\Lambda}{\mu} - (\mu + \delta_i + \gamma_1 - p_i)\right) I_i, \\
&\leq \sum_{i=1}^{n} (\mu + \delta_i + \gamma_i - p_i)\left(\mathcal{T}_{0,i} - 1\right), \\
&\leq 0.
\end{aligned}$$

Now we consider the set contained in Ω where $\dot{V} = 0$.

To have $\dot{V} = 0$ in Ω this implies that for each index

$$I_i = 0 \text{ or } S = \bar{S}_i$$

We associate to each subset $\mathcal{I}$ of indices, a point defined by if $j \notin \mathcal{I}$ then $\mathcal{I}_j = 0$ and if $i \in \mathcal{I}$ then $S = \bar{S}_i$ and for any couple (i, k) in $\mathcal{I}^2$ we assume $\bar{S}_i = \bar{S}_k$. For each kind of subset and condition we have a solution of $\dot{V} = 0$. All these solutions constitute a set $\mathcal{E}$. We state, from the relation (3), that our condition implies $\mathcal{T}_{0,i} = \mathcal{T}_{0,k}$. Then we only consider the subset of indices $\mathcal{I}$ such that for any couple $(j, k) \in \mathcal{I}^2$, $\mathcal{T}_{0,j} = \mathcal{T}_{0,k}$. Any subset of this kind gives a solution in $\mathcal{E}$.

But now we consider the greatest invariant set contained in Ω and in $\mathcal{E}$.

A trajectory starting from one of this point is given by

$$\dot{S} = \Lambda - \bar{S}_i \sum_{i \in \mathcal{I}} \beta_i I_i + \sum_{i \in \mathcal{I}} (\gamma_i - p_i) I_i - \mu \bar{S}_i.$$

We state that for any solution in $\mathcal{E}$ we have $\dot{I}_i = 0$.
By invariance $\dot{S} = 0$, hence

$$\Lambda - \mu \frac{S^*}{\mathcal{T}_{0,i}} = \sum_{i \in \mathcal{I}} \left(\beta_i \frac{S^*}{\mathcal{T}_{0,i}} - (\gamma_i - p_i) \right) I_i = \sum_{i \in \mathcal{I}} (\mu + \delta_i) I_i.$$

Now we recall that $\mathcal{T}_0 \leq 1$, hence each $\mathcal{T}_{0,i} \leq 1$, which implies $\Lambda - \mu \frac{S^*}{\mathcal{T}_{0,i}} \leq 0$.

If $\mathcal{T}_{0,i} < 1$, the preceding inequality cannot be satisfied in the nonnegative orthant. Finally, we see that our set of indices $\mathcal{I}$ is such that $\mathcal{T}_{0,i} = 1$ for any index in $\mathcal{I}$. But this implies again $I_i = 0$ by invariance. Then the only invariant set contained in Ω, such that $\dot{V} = 0$, is the DFE. This proves, by the results of [21, 20], that the DFE is globally asymptotically stable in Ω. Since Ω is an attracting set the stability is global in the nonnegative orthant.

The last assertion implying $\mathcal{R}_0 > 1$, this is a consequence of the results of [12].

■

Remark 3.1

We prove the result without any additional hypotheses on the equality of the $\mathcal{T}_{0,i}$.

3.2 Global stability and competitive exclusion

In this section we assume $\mathcal{R}_0 > 1$ or equivalently $\mathcal{T}_0 > 1$. We will assume in the sequel that one strain is maximizing its threshold. In other words there is a strain (we can give him the index 1) such that for any $i > 1$ we have

$$\mathcal{T}_{0,1} > \mathcal{T}_{0,i}.$$

With $\mathcal{T}_0 > 1$, let i_0 the last index for which $\mathcal{T}_{0,i} > 1$. Then

$$\mathcal{T}_{0,1} > \mathcal{T}_{0,2} \geq \cdots \mathcal{T}_{0,i_0} > 1 \geq \mathcal{T}_{0,i_0+1} \geq \cdots \mathcal{T}_{0,n}.$$

Theorem 3.2

Under the hypothesis $\mathcal{T}_{0,1} > \mathcal{T}_{0,i}$ satisfied for $i = 2, \cdots, n$, the endemic equilibrium

$$\left(\frac{S^*}{\mathcal{T}_{0,1}}, \frac{\Lambda}{\mu + \delta_1} \left(1 - \frac{1}{\mathcal{T}_{0,1}} \right), 0, \cdots, 0 \right),$$

is globally asymptotically stable on the intersection of the of the nonnegative orthant with the two half-hyperspaces defined by the inequations $S > 0$ and $I_1 > 0$.

Proof:

We denote by $(\bar{S}, \bar{I})$, with $\bar{I} = \left(\frac{\Lambda}{\mu+\delta_1} (1 - \frac{1}{\mathcal{T}_{0,1}}), 0, \cdots, 0 \right) \in \mathbb{R}^n_+$ the endemic equilibrium.
We consider the following Lyapunov function, defined on the intersection of the nonnegative orthant with the half open hyperplane spaces given by the inequations $S > 0$ and $I_1 > 0$:

$$V(S, I) = S - \bar{S} \log S + \frac{\mu + \delta_1}{\mu + \delta_1 + \gamma_1 - p_1} \left(I_1 - \bar{I}_1 \log I_1 \right) + \sum_{i=2}^{n} \left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}} \right)^2 \right) I_i + K. \quad (4)$$

Where K is chosen such that $V(\bar{S},\bar{I})=0$.

$$K=-\bar{S}+\bar{S}\log\bar{S}-\frac{\mu+\delta_1}{\mu+\delta_1+\gamma_1-p_1}\left(\bar{I}_1-\bar{I}_1\log\bar{I}_1\right).$$

Indeed this function, on the considered domain, is a positive definite Lyapunov function. To sustain that claim, we must prove that the coefficients of I_i are positive. For this issue we use the following relation

$$\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}=\frac{\frac{\beta_i\Lambda}{\mu(\mu+\delta_i+\gamma_i-p_i)}}{\frac{\beta_1\Lambda}{\mu(\mu+\delta_1+\gamma_1-p_1)}}=\frac{\beta_i\bar{S}}{\mu+\delta_i+\gamma_i-p_i}.$$

From this relation, we deduce the inequality

$$\frac{\gamma_i-p_i}{\beta_i\bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2=\frac{\gamma_i-p_i}{\mu+\delta_i+\gamma_i-p_i}\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}<1. \tag{5}$$

Using the endemic relation $\Lambda=\beta_1\bar{S}\bar{I}_1-(\gamma_1-p_1)\bar{I}_1+\mu\bar{S}$, the derivative of V along trajectories of system (1) is

$$\dot{V}=\left(1-\frac{\bar{S}}{S}\right)\dot{S}+\frac{\mu+\delta_1}{\mu+\delta_1+\gamma_1-p_1}\left(1-\frac{\bar{I}_1}{I_1}\right)I_1+\underbrace{\sum_{i=2}^{n}\left(1-\frac{\gamma_i-p_i}{\beta_i\bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right)\left(\beta_iSI_i-(\mu+\delta_i+\gamma_i-p_i)I_i\right)}_{C}.$$

Which can also be written

$$\begin{aligned}\dot{V} &= \beta_1\bar{S}\bar{I}_1-(\gamma_1-p_1)\bar{I}_1+\mu\bar{S}-\beta_1SI_1-\sum_{i=2}^{n}\beta_iSI_i+(\gamma_1-p_1)I_1+\sum_{i=2}^{n}(\gamma_i-p_i)I_i-\mu S\\ &\quad-\frac{\bar{S}}{S}\left(\beta_1\bar{S}\bar{I}_1-(\gamma_1-p_1)\bar{I}_1+\mu\bar{S}-\beta_1SI_1-\sum_{i=2}^{n}\beta_iSI_i+(\gamma_1-p_1)I_1+\sum_{i=2}^{n}(\gamma_i-p_i)I_i-\mu S\right)\\ &\quad+\frac{\mu+\delta_1}{\mu+\delta_1+\gamma_1-p_1}\left[\beta_1SI_1-(\mu+\delta_1+\gamma_1-p_1)I_1-\beta_1S\bar{I}_1+(\mu+\delta_1+\gamma_1-p_1)\bar{I}_1\right]+C.\\ &= \left[\mu\bar{S}+(\mu+\delta_1)\bar{I}_1+(\gamma_1-p_1)I_1\right]\left(2-\frac{S}{\bar{S}}-\frac{\bar{S}}{S}\right)+\\ &\qquad\sum_{i=2}^{n}\left(\gamma_i-p_i-\beta_iS-\frac{\bar{S}}{S}(\gamma_i-p_i-\beta_iS)\right)I_i+C.\end{aligned} \tag{6}$$

Finally we decompose $\dot{V}$ in the sum of three expressions :

$$\dot{V}=\left[\mu\bar{S}+(\mu+\delta_1)\bar{I}_1+(\gamma_1-p_1)I_1\right]\left(2-\frac{S}{\bar{S}}-\frac{\bar{S}}{S}\right)+\sum_{i=2}^{n}\left(\gamma_i-p_i-\beta_iS-\frac{\bar{S}}{S}(\gamma_i-p_i-\beta_iS)\right)I_i+C.$$

By using the relation

$$\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}} = \frac{\frac{\beta_i \Lambda}{\mu(\mu+\delta_i+\gamma_i-p_i)}}{\frac{\beta_1 \Lambda}{\mu(\mu+\delta_1+\gamma_1-p_1)}} = \frac{\beta_i \bar{S}}{\mu+\delta_i+\gamma_i-p_i}.$$

We have:

$$C = \sum_{i=2}^{n} \left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right) \left[\beta_i S I_i - (\mu+\delta_i+\gamma_i-p_i) I_i\right].$$

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Which can be written

$$C = \sum_{i=2}^{n} \left(\beta_i S - \beta_i \bar{S} \left(\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right) - (\gamma_i - p_i)\frac{S}{\bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2 + (\gamma_i - p_i)\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)\right) I_i. \tag{7}$$

We call B the sum of the last two terms in $\dot{V}$, i.e.,

$$B = \sum_{i=2}^{n} \left(\gamma_i - p_i - \beta_i S - \frac{\bar{S}}{S}(\gamma_i - p_i - \beta_i S)\right) I_2 + C.$$

Then, using relation (7)

$$\begin{aligned} B = &\sum_{i=2}^{n} \left[\gamma_i - p_i - \beta_i\, S - \frac{\bar{S}}{S}(\gamma_i - p_i) + \beta_i\, \bar{S}\right] I_i \\ &+ \sum_{i=2}^{n} \left[\beta_i S - \beta_i \bar{S} \left(\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right) - (\gamma_i - p_i)\frac{S}{\bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2 + (\gamma_i - p_i)\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)\right] I_i. \end{aligned}$$

Which is

$$\begin{aligned} B = \sum_{i=2}^{n} \Bigg[&(\gamma_i - p_i) + \beta_i\, \bar{S} - \frac{\bar{S}}{S}(\gamma_i - p_i) - \beta_i \bar{S} \left(\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right) - (\gamma_i - p_i)\frac{S}{\bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2 \\ &+(\gamma_i - p_i)\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)\Bigg] I_i \end{aligned}$$

Using the inequality between the geometric and arithmetic mean we have

$$-\frac{\bar{S}}{S}(\gamma_i - p_i) - (\gamma_i - p_i)\frac{S}{\bar{S}} \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2 \leq -2\sqrt{(\gamma_i - p_i)^2 \left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2}$$

Hence, with the hypothesis $\gamma_i \geq p_i$, from the preceding inequality we deduce

$$\begin{aligned} B \quad &\leq \sum_{i=2}^{n} \left[-2(\gamma_i - p_i)\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}} + \gamma_i - p_i + \beta_i \bar{S} - \beta_i \bar{S}\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}} + (\gamma_i - p_i)\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right] I_i \\ &\leq \sum_{i=2}^{n} \left[(\gamma_i - p_i)\left(1 - \frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right) + \beta_i \bar{S}\left(1 - \frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right)\right] I_i. \end{aligned}$$

Since $\gamma_i - p_i \leq \mu + \gamma_i + \delta_i - p_i = \beta_i \bar{S} \dfrac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}$, with $\dfrac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}} > 1$, we can write, for each index $i = 2, \cdots, n$, the following inequalities

$$\left[(\gamma_i - p_i)\left(1 - \frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right) + \beta_i \bar{S}\left(1 - \frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right)\right] I_i \leq \left(\beta_i \bar{S}\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\left(1 - \frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right) + \beta_i \bar{S}\left(1 - \frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right)\right) I_i = 0.$$

This relation proves that $B \leq 0$. Then $\dot{V}$ is bounded by the first expression

$$\dot{V} \leq \left[\mu\bar{S} + (\mu + \delta_1)\bar{I}_1 + (\gamma_1 - p_1)I_1\right]\left(2 - \frac{S}{\bar{S}} - \frac{\bar{S}}{S}\right).$$

Since $\gamma_1 \geq p_1$, using again the inequality between the arithmetic and geometric means, we obtain $\dot{V} \leq 0$.

By Lyapunov's theorem this proves the stability of the endemic equilibrium.

To prove the asymptotic stability, we will use LaSalle's principle [20, 21, 15]. We recall the expression of $\dot{V}$

$$\dot{V} = \left(\mu\bar{S} + (\mu + \delta_1)\bar{I}_1 + (\gamma_1 - p_1)I_1\right)\left(2 - \frac{S}{\bar{S}} - \frac{\bar{S}}{S}\right) + \sum_{i=2}^{n}\left(\gamma_i - p_i - \beta_i S - \frac{\bar{S}}{S}(\gamma_i - p_i - \beta_i S)\right) I_i + C. \quad (8)$$

We have to find the points $(S, I_1, \cdots, I_n)$ for which $\dot{V} = 0$

We have seen that V is the sum of three nonpositive quantities. The first term, a positive definite function of S, is zero if and only if $S = \bar{S}$.

The second term, with $S = \bar{S}$, is equal to zero since

$$\sum_{i=2}^{n}\left(\gamma_i - p_i - \beta_i S - \frac{\bar{S}}{S}(\gamma_i - p_i - \beta_i S)\right) I_i\Bigg|_{S=\bar{S}} = 0.$$

Ê

Finally,

$$\begin{aligned} C &= \sum_{i=2}^{n}\left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right)(\beta_i S I_i - (\mu + \delta_i + \gamma_i - p_i)I_i) \\ &= \sum_{i=2}^{n}\left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right)(\beta_i \bar{S} - (\mu + \delta_i + \gamma_i - p_i))\, I_i \\ &= \sum_{i=2}^{n}\left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right)\left(\beta_i \bar{S} - \beta_i \bar{S}\frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right) I_i \\ &= \sum_{i=2}^{n}\underbrace{\left(1 - \frac{\gamma_i - p_i}{\beta_i \bar{S}}\left(\frac{\mathcal{T}_{0,i}}{\mathcal{T}_{0,1}}\right)^2\right)}_{>0}\ \beta_i \bar{S}\ \underbrace{\left(1 - \frac{\mathcal{T}_{0,1}}{\mathcal{T}_{0,i}}\right)}_{<0} I_i. \end{aligned}$$

Then $C = 0$ if and only each $I_i = 0$ for $i = 2, \cdots, n$.

We found that $\dot{V} = 0$ implies $S = \bar{S}$ and $I_i = 0$ for $i = 2, \cdots, n$. We conclude by the Lasalle's principle, that the greatest invariant set is reduced to the endemic equilibrium.

This ends the proof of the theorem .

■

For any strain, we have defined $\mathcal{T}_{0,i} = \dfrac{\beta_i \Lambda}{\mu(\mu + \gamma_i + \delta_i - p_i)} = \dfrac{\Lambda}{\mu \bar{S}_i}$.
We showed, that the winner strain maximizes the threshold $\mathcal{T}_{0,i}$, thus minimizes $\bar{S}_i$. This result is analogous to those obtained in [17]. This can be interpreted like a pessimization principle [11, 30] .

4 SIR model

In this section, we consider a SIR model where a proportion of infected can recovered and have total immunity for all strains. For this model, we have the following system:

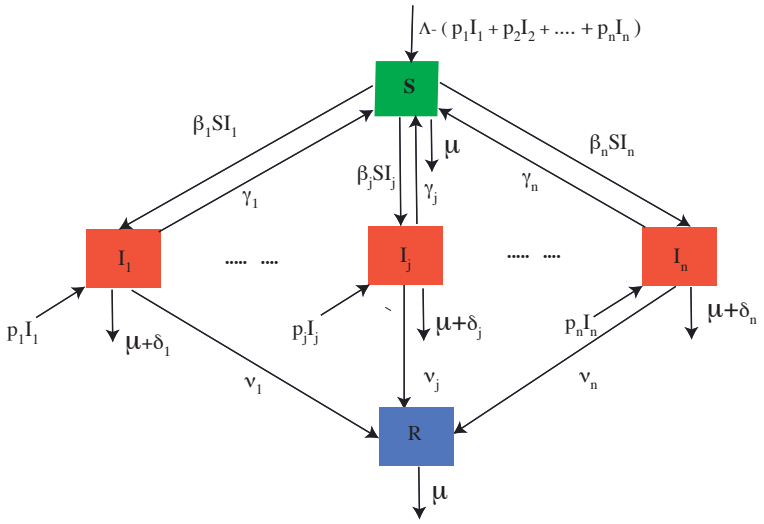

Figure 2: A SIR model with n strains

$$\begin{cases} \dot{S} = \Lambda - \sum_{i=1}^{n} \beta_i S I_i + \sum_{i=1}^{n} (\gamma_i - p_i) I_i - \mu S, \\ \dot{I}_i = \beta_i S I_i - (\mu + \delta_i + \nu_i + \gamma_i - p_i) I_i \quad i = 1, \ldots, n \\ \dot{R} = \sum_{i=1}^{n} \nu_i I_i - \mu R \end{cases} \tag{9}$$

Since R is not present in the two fisrt equation, we have a triangular system. We are in a compact domain. By Vidyasagar's theorem [34], to study stability, it is sufficient to study the system composed by the two first equations. Then the above results are valid for the same Lyapunov functions by replacing δ_i by $\delta_i' = \delta_i + \nu_i$.

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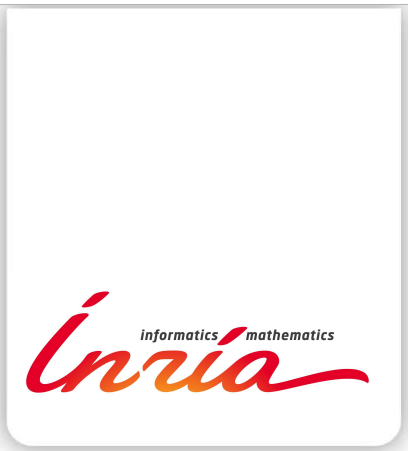

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